Lecture Notes, Lectures 12 -13

5. 1 Household Consumption Sets and Preferences

H, i= 1, 2, ..., #H

$i \in H$, $X^i \subseteq R^N_+$, $u^i : X^i \rightarrow R$ (fully represents $\succsim_i$), $r^i \in R^N_+$,

$1 \geq \alpha^{ij} \geq 0$ for each $j \in F$.

$x \in X^i$, $x=(x_1, x_2, ..., x_N)$

<u>Consumption Sets</u>

(C.I) X^i is closed and nonempty.

(C.II) $X^i \subseteq R^N_+$. X^i is bounded below and unbounded above.

(C.III) X^i is convex.

X^i may be R_+^N.
$X = \sum_{i \in H} X^i$.

Preferences

x, y $\in X^i$, "x $\succsim_i$ y" is read "x is preferred or indifferent to y (according to i)."

Utility Function

Let u^h: $X^h \to R$. Then u^h is a utility function.

Definition: We will say that the utility function $u^h(\bullet)$ <u>represents</u> the preference order $\succsim_h$ if for all x, y $\in X^h$,

$u^h(x) \geq u^h(y)$ if and only if $x \succsim_h y$. This implies that

$u^h(x) > u^h(y)$ if and only if $x \succsim_h y$ and not $[y \succsim_h x]$.

We will assume there is $u^i: X^i \to \mathbf{R}$ so that $u^i(\ )$ represents $\succsim_i$

.

Read $u^i(x) \geq u^i(y)$ wherever you see $x \succsim_i y$.

<u>Weak monotonicity</u>

(C.IV) (Weak Monotonicity) Let x, y $\in X^i$, with
x >> y, (that is, $x_i > y_i, i = 1, \ldots, N$) .

Then $u^i(x) > u^i(y)$.

Continuity

(C.V) (Continuity) For every $x^o \in X^i$ the sets

$A^i(x^o) = \{x | x \in X^i, x \succsim_i x^0\}$ and

$G^i(x^o) = \{x | x \in X^i, x^0 \succsim_i x\}$ are closed. That is, the inverse images of closed subsets of R under u(•) are closed.

Equivalently: $u^i(\ )$ is a continuous function.

Continuity of u^i allows us to use Corollary 2.2. What does the continuity assumption rule out? Lexicographic preferences provide an example of discontinuous preferences (which cannot be represented by a utility function; certainly not by a continuous utility function).

Example (Lexicographic preferences): $>_L$ $x = (x_1, x_2, ..., x_N)$, $y = (y_1, y_2, ..., y_N)$.

$x >_L y$ if $x_1 > y_1$, or

if $x_1 = y_1$, and $x_2 > y_2$, or

if $x_1 = y_1$, and $x_2 = y_2$, and $x_3 > y_3$, and so forth ....

$x \sim_L y$ if $x = y$.

Strict Convexity of Preferences

(C.VII) (strict convexity of preferences):

$u^i(x) \geq u^i(y)$, $x \neq y$, $0 < \alpha < 1$, implies

$$u^i(\alpha x + (1-\alpha)y) > u^i(y).$$

5.3 Choice and Boundedness of Budget Sets, $\tilde{B}^i(p)$

Definition: x is an <u>attainable</u> aggregate consumption if $y + r \geq x \geq 0$ where $y \in Y$ and $r \in R_+^N$ is the economy's initial resource endowment, so that y is an attainable production plan. Note that the set of attainable consumptions is bounded under P.II , P.III, P.V, P.VI.

Choose $c \in R_+$ so that $|x| < c$ (a strict inequality) for all attainable consumptions x. Choose c sufficiently large that $X^i \cap \{x \mid x \in R^N, c \geq |x|\} \neq \phi$.

$\tilde{M}^i(p)$ represents i's income as a function of p. We do not need precisely to specify $\tilde{M}^i(p)$ at this point.

When we do, income will be characterized as the value of the household endowment plus the value of the household share of firm profits = $p \cdot r^i + \Sigma_j \alpha^{ij} \tilde{\pi}^j(p)$.

$$\tilde{B}^i(p) = \left\{ x \mid x \in R^N,\ p \cdot x \leq \tilde{M}^i(p) \right\} \cap \{ x \mid |x| \leq c \}.$$

$$\tilde{D}^i(p) \equiv \{ x \mid x \in \tilde{B}^i(p) \cap X^i, \text{ x maximizes } u^i(y) \text{ for all } y \in \tilde{B}^i(p) \cap X^i \}$$

$$\tilde{D}(p) = \sum_{i \in H} \tilde{D}^i(p) .$$

Lemma 5.1: $\tilde{B}^i(p)$ is a closed and bounded (compact) set.

Lemma 5.2: Let $\tilde{M}^i(p)$ be homogeneous of degree 1. Then $\tilde{B}^i(p)$ and $\tilde{D}^i(p)$ are homogeneous of degree 0.

$$P \equiv \{ p \mid p \in R^N , p_i \geq 0, i = 1, 2, 3, ..., N, \sum_{i=1}^{N} p_i = 1 \}$$

<u>Positivity of Income</u>

(C.VIII) $\widetilde{M}^i(p) > \min_{x \in X^i \cap \{y | y \in R^N, c \geq |y|\}} p \cdot x \geq 0$ for all $p \in P$.

Example (The Arrow Corner): This example demonstrates the importance of (C.VIII). (C.VIII) is not fulfilled in the example resulting in discontinuous demand.

$$X^i = R_+^2$$

$$r^i = (1, 0)$$

$$\widetilde{M}^i(p) = p \cdot r^i .$$

$$p^0 = (0,\ 1)\,.$$

$$\tilde{B}^i(p^0) \cap X^i = \{(x,y) | c \geq x \geq 0,\ y = 0\}$$

$$p^\nu = \left(\frac{1}{\nu},\ 1 - \frac{1}{\nu}\right)\,.\ \ \mathrm{p}^\nu \to \mathrm{p}^0\,.$$

$$\tilde{B}^i(p^\nu) \cap X^i = \left\{(x,y) | p^\nu \cdot (x,y) \leq \frac{1}{\nu}, (x,y) \geq 0, c \geq |(x,y)| \geq 0\right\}$$

$(c,\ 0) \in \tilde{B}^i(p^0)$ but there is no sequence

$(x^\nu,\ y^\nu) \in \tilde{B}^i(p^\nu)$ so that $(x^\nu,\ y^\nu) \to (c,\ 0)$.

For any sequence $(x^\nu,\ y^\nu) \in \tilde{B}^i(p^\nu)$ so that

$(x^\nu,\ y^\nu) = \tilde{D}^i(p^\nu)$, $(x^\nu,\ y^\nu)$ will converge to some $(x^*,0)$

where $0 \leq x^* \leq 1$. We may have (c, 0) = $\tilde{D}^i(p^0)$. Hence $\tilde{D}^i(p)$ need not be continuous at p^0. This completes the example.

5.4 Demand behavior under strict convexity

Theorem 5.2: Assume C.I - C.V, C.VII, C.VIII. Let $\tilde{M}^i$(p) be a continuous function for all p ∈ P. Then $\tilde{D}^i(p)$ is a well-defined, point-valued, continuous function for all p ∈ P.

Proof: Well defined: Compactness of $\tilde{B}^i$(p) ∩ X^i and continuity of $u^i(\bullet)$.

Unique (point valued): Strict convexity of preferences, C.VII.

Continuous

C.VIII ⇒ $\tilde{M}^i(p) > 0$ for all p ∈ P.

Let $p^\nu \in P$, $\nu = 1, 2, 3, \ldots$, $p^\nu \to p^o$. Show $\tilde{D}^i(p^\nu) \to \tilde{D}^i(p^o)$. $\tilde{D}^i(p^\nu)$ is a sequence in a compact set.

Without loss of generality take a convergent subsequence,

$\tilde{D}^i(p^\nu) \to x^o$. We must show that $x^o = \tilde{D}^i(p^o)$. Proof by contradiction.

Define $\hat{x} = \underset{x \in X^i \cap \{y | y \in R^N, c \geq |y|\}}{\arg\min} p^o \cdot x$.

$p^o \cdot \tilde{D}^i(p^o) > p^o \cdot \hat{x}$ (by C.VIII).

Let $\alpha^\nu = \min\left[1, \dfrac{\tilde{M}^i(p^\nu) - p^\nu \cdot \hat{x}}{p^\nu \cdot (\tilde{D}^i(p^o) - \hat{x})}\right]$. For ν large, α^ν is well defined. $0 \leq \alpha^\nu \leq 1$. $\alpha^\nu \to 1$. Let $w^\nu = (1 - \alpha^\nu)\,\hat{x} + \alpha^\nu\, \tilde{D}^i(p^o)$.

$w^\nu \to \tilde{D}^i(p^o)$ and $w^\nu \in \tilde{B}^i(p^\nu) \cap X^i$. Suppose

$x^o \neq \tilde{D}^i(p^o)$. Then $u^i(\tilde{D}^i(p^o)) > u^i(x^o)$. But for ν large, $u^i(w^\nu) > u^i(\tilde{D}^i(p^\nu))$ by continuity of u^i and the convergence of $w^\nu \to \tilde{D}^i(p^o)$, $\tilde{D}^i(p^\nu) \to x^o$. This is a contradiction, since $\tilde{D}^i(p^\nu)$ maximizes $u^i(\cdot)$ in $\tilde{B}^i(p^\nu) \cap X^i$. QED

Lemma 5.3: Assume C.I - C.V, C.VII, C.VIII. Then $p \cdot \tilde{D}^i(p) \leq \tilde{M}^i(p)$. Further, if $p \cdot \tilde{D}^i(p) < \tilde{M}^i(p)$ then $|\tilde{D}^i(p)| = c$.

Proof: Budget or length is a binding constraint --- if not budget, then length.